

POWERS OF COMPLETE INTERSECTIONS: GRADED BETTI NUMBERS AND APPLICATIONS

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ABSTRACT. Let $I = (F_1, \dots, F_r)$ be a homogeneous ideal of the ring $R = k[x_0, \dots, x_n]$ generated by a regular sequence of type (d_1, \dots, d_r) . We give an elementary proof for an explicit description of the graded Betti numbers of I^s for any $s \geq 1$. These numbers depend only upon the type and s . We then use this description to: (1) write H_{R/I^s} , the Hilbert function of R/I^s , in terms of $H_{R/I}$; (2) verify that the k -algebra R/I^s satisfies a conjecture of Herzog-Huneke-Srinivasan; and (3) obtain information about the numerical invariants associated to sets of fat points in \mathbb{P}^n whose support is a complete intersection or a complete intersection minus a point.

INTRODUCTION

In this paper we give an explicit description of the graded Betti numbers of a power of a complete intersection and provide some applications of this result.

It is well known that the graded minimal free resolution of a homogeneous complete intersection $I = (F_1, \dots, F_r) \subseteq R = k[x_0, \dots, x_n]$ is given by the Koszul resolution. Furthermore, the graded Betti numbers of I depend only upon the type (d_1, \dots, d_r) where $d_i = \deg F_i$ (cf. Theorem 1.1). However, the description of the graded Betti numbers of a power of a complete intersection does not enjoy the same level of familiarity.

It has long been known that I^s , the power of a complete intersection, can be realized as the ideal generated by the maximal minors of a $s \times (s+r-1)$ matrix with entries consisting of the F_i 's. Since I^s is an example of a determinantal ideal, a minimal free resolution of I^s can be obtained by using the Eagon-Northcott complex. However, the maps in this resolution may not be degree preserving, so the graded Betti numbers cannot be extracted.

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Alternatively, a resolution of I^s can be found in [2] (see [17, Theorem 2.1] for a more relevant formulation). Implicit in [17] is the fact that the minimal resolution of I^s is a graded minimal resolution of I^s provided I is homogeneous. The graded Betti numbers of I^s are a consequence of this result, but to the best of our knowledge, they have never been explicitly described. This probably accounts for the lack of familiarity mentioned above for these invariants.

Our main result (Theorem 2.1) is to give an explicit description for the graded Betti numbers of I^s . As we shall show, these numbers can be computed directly from the type (d_1, \dots, d_r) and s . To further differentiate our result from past work, we give an elementary proof which avoids the machinery used in [2, 17]. We relate the ideals I^s , I^{s-1} and $(F_2, \dots, F_r)^s$ through a basic double link. Then, by using a mapping cone construction and induction on the tuple (r, s) , we obtain the graded Betti numbers of I^s .

As a corollary, we express H_{R/I^s} , the Hilbert function of R/I^s , as a function of $H_{R/I}$. Like the graded Betti numbers of I^s , we were unable to find in the literature an expression for H_{R/I^s} . This omission is also noted in [9, Page 799].

The final three sections are devoted to applications of Theorem 2.1. In §3, we verify that the k -algebra R/I^s satisfies a conjecture of Herzog-Huneke-Srinivasan which relates the multiplicity $e(R/I^s)$ to the shifts in the graded minimal free resolution of R/I^s .

In §4, we use Theorem 2.1 to study sets of fat points $Z \subseteq \mathbb{P}^n$ whose support is a complete intersection. When Z is homogeneous (all the points have the same multiplicity), then Theorem 2.1 gives the graded minimal free resolution of I_Z . When Z is not homogeneous, we obtain partial information about the invariants associated to I_Z .

In §5, we investigate sets of fat points in \mathbb{P}^n whose support is a complete intersection minus a point. We give examples to show different constructions of the underlying complete intersection, e.g. whether or not the defining forms are irreducible or reducible, may result in different numerical invariants, thus implying extra hypotheses on the support are needed. We therefore study such schemes under the extra condition that the underlying complete intersection can “split” into smaller complete intersections. This restriction allows us to use Theorem 2.1 to obtain bounds on the least degree of a form passing through the scheme and the regularity index. This section extends some of the results of [3, 4, 13] which studied this question in \mathbb{P}^2 .

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1. COMPLETE INTERSECTIONS

Let k denote an infinite field with $\text{char}(k) = 0$, and set $R = k[x_0, \dots, x_n]$. A homogeneous ideal $I = (F_1, \dots, F_r) \subseteq R$ is a *complete intersection* if F_1, \dots, F_r form a regular sequence on R . If $\deg F_i = d_i$, then we say that I is a complete intersection of type (d_1, \dots, d_r) . Since F_1, \dots, F_r are homogeneous, any permutation of their order again results in a regular sequence. Thus, we can assume that $d_1 \leq d_2 \leq \dots \leq d_r$. The graded Betti numbers in the graded minimal free resolution of I depend only upon the type as described below.

Theorem 1.1 (Koszul Resolution). *Let $I \subseteq R$ be a complete intersection of type (d_1, \dots, d_r) . Then the graded minimal free resolution of I has the form*

$$0 \rightarrow \mathcal{F}_{r-1} \rightarrow \mathcal{F}_{r-2} \rightarrow \dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow I \rightarrow 0$$

where

$$\mathcal{F}_j = \bigoplus_{1 \leq i_1 < i_2 < \dots < i_{j+1} \leq r} R(-d_{i_1} - d_{i_2} - \dots - d_{i_{j+1}}) \text{ for } j = 0, \dots, r-1.$$

Some of the properties of the ideal I^s , where I is a complete intersection and $s \in \mathbb{N}^+$, are given below.

Theorem 1.2. *Let $I = (F_1, \dots, F_r)$ be an ideal of $R = k[x_0, \dots, x_n]$ generated by a regular sequence of length $r \leq n+1$, and let \mathcal{A} be the $s \times (s+r-1)$ matrix*

$$\mathcal{A} = \begin{pmatrix} F_1 & F_2 & \dots & F_r & 0 & 0 & 0 & \dots & 0 \\ 0 & F_1 & F_2 & \dots & F_r & 0 & 0 & \dots & 0 \\ 0 & 0 & F_1 & F_2 & \dots & F_r & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & F_1 & F_2 & \dots & F_r \end{pmatrix}.$$

Then

- (i) $I^s = I_s(\mathcal{A})$, the ideal generated by the $s \times s$ maximal minors of \mathcal{A} .
- (ii) for each $s \geq 1$, the ideal I^s is perfect of grade r .
- (iii) the minimal free resolution of I^s is given by the Eagon-Northcott complex constructed from \mathcal{A} .

Proof. Statement (i) is in [5] following Remark 2.13. Statement (ii) is [5, Proposition 2.14]. Since the grade of $I_s(\mathcal{A})$ is r , the Eagon-Northcott complex constructed from \mathcal{A} gives a minimal free resolution of $R/I_s(\mathcal{A})$ by [8, Theorem 5.2]. \square

Although the Eagon-Northcott complex gives a minimal resolution of I^s , the maps in the resolution may not be degree preserving, and thus, we cannot use this resolution to compute the graded Betti numbers except under extra hypotheses on the type, e.g., [1]. To see this, note that the matrix in the above theorem defines a map $R^{s+r-1} \xrightarrow{A} R^s$. This map may fail to be a homogeneous map, e.g., if the degrees of the F_i 's fail to be an arithmetic progression, and thus the maps in Eagon-Northcott complex are not homogeneous. However, we can use the Eagon-Northcott resolution of I^s to determine the ranks of the modules in the resolution, thus allowing us to count the number of generators of each syzygy module.

Corollary 1.3. *With the hypotheses as in the previous theorem, let*

$$0 \rightarrow \mathcal{F}_{r-1} \rightarrow \mathcal{F}_{r-2} \rightarrow \cdots \rightarrow \mathcal{F}_0 \rightarrow I^s \rightarrow 0$$

be the minimal free resolution of I^s given by the Eagon-Northcott complex. Then

$$\mathrm{rk} \mathcal{F}_i = \binom{r+s-1}{s+i} \binom{s-1+i}{i} \text{ for } i = 0, \dots, r-1.$$

The following lemma relates I^s to a complete intersection with one less generator.

Lemma 1.4. *Let F_1, \dots, F_r be a regular sequence in R with $\deg F_i = d_i$. Set $I = (F_1, \dots, F_r)$ and $J = (F_2, \dots, F_r)$. Then for each positive integer s ,*

$$I^s = J^s + F_1 \cdot I^{s-1}.$$

Furthermore, we have the following short exact sequence

$$(1.1) \quad 0 \longrightarrow J^s(-d_1) \longrightarrow I^{s-1}(-d_1) \oplus J^s \longrightarrow J^s + F_1 \cdot I^{s-1} = I^s \longrightarrow 0.$$

Proof. The equality of ideals is immediate. The short exact sequence is from [16, Lemma 4.8]. \square

2. GRADED BETTI NUMBERS OF POWERS OF COMPLETE INTERSECTIONS

Let $I = (F_1, \dots, F_r)$ be a homogeneous complete intersection of $R = k[x_0, \dots, x_n]$. For each $s \in \mathbb{N}^+$ we describe how the graded Betti numbers in the graded minimal free resolution of I^s depend only upon the type (d_1, \dots, d_r) and s . To describe the resolution we introduce the sets

$$\mathcal{M}_{r,s,t} := \left\{ (a_1, \dots, a_r) \in \mathbb{N}^r \mid \begin{array}{l} a_1 + \cdots + a_r = s \text{ and at} \\ \text{least } t \text{ of the } a_i \text{'s are non-zero} \end{array} \right\}$$

for all positive integers r, s , and t . With this notation we have

Theorem 2.1. *Let $I \subseteq R$ be a complete intersection of type (d_1, \dots, d_r) and let $s \in \mathbb{N}^+$. Then the graded minimal free resolution of I^s has the form*

$$0 \rightarrow \mathcal{H}_{r-1} \rightarrow \mathcal{H}_{r-2} \rightarrow \cdots \rightarrow \mathcal{H}_0 \rightarrow I^s \rightarrow 0$$

where

$$\mathcal{H}_0 = \bigoplus_{(a_1, \dots, a_r) \in \mathcal{M}_{r,s,1}} R(-a_1 d_1 - \dots - a_r d_r)$$

and for $i = 1, \dots, r-1$,

$$\mathcal{H}_i = \bigoplus_{l_1=i+1}^r \left[\bigoplus_{l_2=l_1}^r \left[\dots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{(a_1, \dots, a_r) \in \mathcal{M}_{r,s+i,l_i}} R(-a_1 d_1 - \dots - a_r d_r) \right] \dots \right] \right] \right].$$

Proof. The proof is by induction on the tuple (r, s) . If $s = 1$, and $r \leq n+1$ is any positive integer, then the resolution of I is given by the Koszul resolution (Theorem 1.1). The reader can verify that the sets $\mathcal{M}_{r,s,t}$ account for the expected Betti numbers in this case. If $r = 1$, and s is any integer, then I^s is principal, and the result also follows.

So, let (r, s) be a tuple with $1 < r \leq n+1$ and $s \in \mathbb{N}^+$, and assume that the result holds for all ideals of the form $(F_1, \dots, F_{r'})^{s'}$ with $(r', s') <_{lex} (r, s)$ with respect to the lexicographical ordering. Set $I = (F_1, \dots, F_r)$ and $J = (F_2, \dots, F_r)$. The short exact sequence (1.1) of Lemma 1.4 relates I^s, I^{s-1} and J^s :

$$0 \longrightarrow J^s(-d_1) \longrightarrow I^{s-1}(-d_1) \oplus J^s \longrightarrow I^s \longrightarrow 0.$$

Let

$$0 \rightarrow \mathcal{G}_{r-2} \rightarrow \mathcal{G}_{r-3} \rightarrow \dots \rightarrow \mathcal{G}_0 \rightarrow J^s \rightarrow 0 \text{ and}$$

$$0 \rightarrow \mathcal{K}_{r-1} \rightarrow \mathcal{K}_{r-2} \rightarrow \dots \rightarrow \mathcal{K}_0 \rightarrow I^{s-1} \rightarrow 0$$

be the minimal free resolutions of J^s and I^{s-1} , respectively. The mapping cone construction and the exact sequence then gives the following resolution of I^s :

$$(2.1) \quad 0 \rightarrow \mathcal{H}_{r-1} \rightarrow \mathcal{H}_{r-2} \rightarrow \dots \rightarrow \mathcal{H}_0 \rightarrow I^s \rightarrow 0$$

where

$$\mathcal{H}_0 = \mathcal{G}_0 \oplus \mathcal{K}_0(-d_1)$$

$$\mathcal{H}_i = \mathcal{G}_{i-1}(-d_1) \oplus \mathcal{G}_i \oplus \mathcal{K}_i(-d_1) \text{ for } i = 1, \dots, r-2$$

$$\mathcal{H}_{r-1} = \mathcal{G}_{r-2}(-d_1) \oplus \mathcal{K}_{r-1}(-d_1).$$

We verify first that (2.1) is minimal. The resolution has the correct length by Theorem 1.2. To show that the sequence does not split, it is enough to apply induction to show that the rank of each \mathcal{H}_i equals the rank expected by Corollary 1.3. Indeed, for $i = 1, \dots, r-2$ we get

$$\begin{aligned} \text{rk } \mathcal{H}_i &= \text{rk } \mathcal{G}_{i-1} + \text{rk } \mathcal{G}_i + \text{rk } \mathcal{K}_i \\ &= \binom{r+s-2}{s+i-1} \binom{s+i-2}{i-1} + \binom{r+s-2}{s+i} \binom{s+i-1}{i} + \binom{r+s-2}{s+i-1} \binom{s+i-2}{i} \\ &= \binom{r+s-1}{s+i} \binom{s-i-1}{i}. \end{aligned}$$

The proofs that \mathcal{H}_0 and \mathcal{H}_{r-1} have the correct ranks follow similarly, so we have omitted them. It now follows that (2.1) is minimal.

We now compute the graded Betti numbers of I^s by applying the induction hypothesis to J^s and I^{s-1} . Due to the tedious nature of the proof, we shall only show that \mathcal{H}_i has the correct Betti numbers for $i = 2, \dots, r-2$, and omit the similar proofs for $\mathcal{H}_0, \mathcal{H}_1$, and \mathcal{H}_r .

For $i = 2, \dots, r-2$ we have $\mathcal{H}_i = \mathcal{G}_{i-1}(-d_1) \oplus \mathcal{G}_i \oplus \mathcal{K}_i(-d_1)$. The induction hypothesis applied to $\mathcal{G}_{i-1}(-d_1)$ gives

$$\begin{aligned} \mathcal{G}_{i-1}(-d_1) &= \bigoplus_{l_2=i}^{r-1} \left[\bigoplus_{l_3=l_2}^{r-1} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{r-1} \left[\bigoplus_{\mathcal{M}_{r-1,s+i-1,l_i}} R(-a_2 d_2 - \cdots - a_r d_r - d_1) \right] \cdots \right] \right] \right] \\ &= \bigoplus_{l_2=i}^{r-1} \left[\bigoplus_{l_3=l_2}^{r-1} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{r-1} \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i+1} \\ \text{and } a_1=1}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \\ &= \bigoplus_{l_2=i+1}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=1}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right]. \end{aligned}$$

We can write \mathcal{G}_i as

$$\begin{aligned} \mathcal{G}_i &= \bigoplus_{l_1=i+1}^{r-1} \left[\bigoplus_{l_2=l_1}^{r-1} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{r-1} \left[\bigoplus_{\mathcal{M}_{r-1,s+i,l_i}} R(-a_2 d_2 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \\ &= \bigoplus_{l_2=i+1}^{r-1} \left[\bigoplus_{l_3=l_2}^{r-1} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{r-1} \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=0}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \oplus \\ &\quad \vdots \\ &\quad \bigoplus_{l_2=r-1}^{r-1} \left[\bigoplus_{l_3=l_2}^{r-1} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{r-1} \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=0}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right]. \end{aligned}$$

Because the set $\{(a_1, \dots, a_r) \in \mathcal{M}_{r,s+i,r} \mid a_1 = 0\} = \emptyset$, \mathcal{G}_i is equal to

$$\begin{aligned} \mathcal{G}_i &= \bigoplus_{l_2=i+1}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=0}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \oplus \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
& \bigoplus_{l_2=r-1}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=0}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \oplus \\
& \bigoplus_{l_2=r}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1=0}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right].
\end{aligned}$$

For $\mathcal{K}_i(-d_i)$, the induction hypothesis gives

$$\begin{aligned}
\mathcal{K}_i(-d_1) &= \bigoplus_{l_1=i+1}^r \left[\bigoplus_{l_2=l_1}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\mathcal{M}_{r,s+i-1,l_i}} R(-a_1 d_1 - \cdots - a_r d_r - d_1) \right] \cdots \right] \right] \right] \\
&= \bigoplus_{l_2=i+1}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\mathcal{M}_{r,s+i-1,l_i}} R(-a_1 d_1 - \cdots - a_r d_r - d_1) \right] \cdots \right] \right] \right] \oplus \\
&\quad \vdots \\
&\quad \bigoplus_{l_2=r}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\mathcal{M}_{r,s+i-1,l_i}} R(-a_1 d_1 - \cdots - a_r d_r - d_1) \right] \cdots \right] \right] \right].
\end{aligned}$$

We have the following identity:

$$\begin{aligned}
\bigoplus_{\mathcal{M}_{r,s+i-1,l_i}} R(-a_1 d_1 - \cdots - a_r d_r - d_1) &= \bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 \geq 2}} R(-a_1 d_1 - \cdots - a_r d_r) \oplus \\
&\quad \bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i+1} \\ \text{and } a_1 = 1}} R(-a_1 d_1 - \cdots - a_r d_r).
\end{aligned}$$

Because of this identity, we can write $\mathcal{K}_i(-d_1)$ as

$$\begin{aligned}
\mathcal{K}_i(-d_1) &= \bigoplus_{l_2=i+1}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 \geq 2}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \oplus \\
&\quad \bigoplus_{l_2=i+2}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 = 1}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \cdots \right] \right] \right] \oplus
\end{aligned}$$

$$\begin{aligned}
& \left[\bigoplus_{l_2=i+2}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 \geq 2}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \right] \right] \right] \right] \oplus \\
& \vdots \\
& \left[\bigoplus_{l_2=r}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 = 1}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \right] \right] \right] \right] \oplus \\
& \left[\bigoplus_{l_2=r}^r \left[\bigoplus_{l_3=l_2}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{\substack{\mathcal{M}_{r,s+i,l_i} \\ \text{and } a_1 \geq 2}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \right] \right] \right] \right] \right].
\end{aligned}$$

As a consequence, since $\mathcal{H}_i = \mathcal{G}_{i-1}(-d_1) \oplus \mathcal{G}_i \oplus \mathcal{K}_i(-d_1)$, we have

$$\mathcal{H}_i = \bigoplus_{l_1=i+1}^r \left[\bigoplus_{l_2=l_1}^r \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^r \left[\bigoplus_{(a_1, \dots, a_r) \in \mathcal{M}_{r,s+i,l_i}} R(-a_1 d_1 - \cdots - a_r d_r) \right] \right] \right] \right].$$

as desired. \square

Remark 2.2. The minimal resolution of I^s as given in [17, Theorem 2.1] is also graded if I is homogeneous, although it is not explicitly stated as such. For the convenience of the reader, we sketch out how to make the resolution graded. Let F be a free R -module with $\text{rk } F = r$. Let $L_i^s F$ denote the image of the map $d_{i,s} : \bigwedge^i F \otimes S_{s-1} F \longrightarrow \bigwedge^{i-1} F \otimes S_s F$ where $d_{i,s}$ is defined as in [17, Definition 1.1] and SF is the symmetric algebra on F . The resolution of $I^s = (F_1, \dots, F_r)^s$ then has the form:

$$0 \rightarrow L_r^s F \rightarrow L_{r-1}^s F \rightarrow \cdots \rightarrow L_2^s F \rightarrow S_s F \rightarrow I^s \rightarrow 0.$$

As explained in [17, Page 152], a basis for $L_i^s F$ for each i can be identified with the standard tableau of the form

$$C = \begin{array}{|c|c|c|c|} \hline t_1 & t_2 & \cdots & t_i \\ \hline p_2 & & & \\ \hline \vdots & & & \\ \hline p_s & & & \\ \hline \end{array}$$

with $1 \leq t_1 < t_2 < \cdots < t_i \leq r$ and $t_1 \leq p_2 \leq \cdots \leq p_s \leq r$. To make this resolution graded, assign a degree to each tableau basis element as follows:

$$\deg C := d_{t_1} + d_{t_2} + \cdots + d_{t_i} + d_{p_2} + \cdots + d_{p_s}$$

where d_{i_j} is the i_j th element of (d_1, \dots, d_r) .

If J is a homogeneous ideal of R , then the *Hilbert function* of R/J is $H_{R/J}(i) := \dim_k(R/J)_i$. As a corollary of Theorem 2.1 we obtain a formula for the Hilbert function of R/I^s . We define

$$\mathcal{L}_{r,s} := \{(a_1, \dots, a_r) \in \mathbb{N}^r \mid a_1 + \cdots + a_r \leq s-1\}.$$

Corollary 2.3. *With the hypotheses as in Theorem 2.1,*

$$H_{R/I^s}(i) = \sum_{(a_1, \dots, a_r) \in \mathcal{L}_{r,s}} H_{R/I}(i - a_1 d_1 - \cdots - a_r d_r)$$

where $H_{R/I}$ is the Hilbert function of R/I .

Remark 2.4. Corollary 2.3 can also be proved directly by using Lemma 1.4 and an induction proof similar to the proof of Theorem 2.1. The fact that H_{R/I^s} depends only upon $H_{R/I}$ was first observed in [7].

3. APPLICATION: A CONJECTURE OF HERZOG, HUNEKE, AND SRINIVASAN

As an application of Theorem 2.1, we can verify a special case of a conjecture of Herzog, Huneke, and Srinivasan [14]. The following relation between the multiplicity $e(R/I)$ of R/I and the degrees of the syzygies of I is conjectured to hold:

Conjecture 3.1. *Let R/I be a Cohen-Macaulay k -algebra with resolution of the form*

$$0 \longrightarrow \bigoplus_{j=1}^{b_r} R(-d_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} R(-d_{1j}) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Set $m_i = \min\{d_{ij} \mid j = 1, \dots, b_i\}$ and $M_i = \max\{d_{ij} \mid j = 1, \dots, b_i\}$. Then

$$\frac{\prod_{i=1}^r m_i}{r!} \leq e(R/I) \leq \frac{\prod_{i=1}^r M_i}{r!}.$$

By using Theorem 2.1, powers of complete intersections can be added to the list of ideals in [11, 15] that satisfy the conjecture. The case $s = 1$ is also found in [14].

Proposition 3.2. *Let I be a complete intersection of R of type (d_1, \dots, d_r) , and let s be any positive integer. Then Conjecture 3.1 is true for R/I^s .*

Proof. We can assume $d_1 \leq \dots \leq d_r$. The resolution of I^s given in Theorem 2.1 implies $m_i = sd_1 + d_2 + \dots + d_i$ and $M_i = d_{r-i+1} + \dots + d_{r-1} + sd_r$ for $i = 1, \dots, r$. Because $e(R/I^s) = \binom{s+r-1}{r} d_1 \dots d_r$ and

$$\frac{\prod_{i=1}^r m_i}{r!} \leq \prod_{i=1}^r \frac{s+(i-1)}{i} d_i = e(R/I^s) = \prod_{i=1}^r \frac{s+(i-1)}{i} d_{r-i+1} \leq \frac{\prod_{i=1}^r M_i}{r!},$$

R/I^s satisfies the conjecture. \square

4. APPLICATION: FAT POINTS WITH SUPPORT ON A COMPLETE INTERSECTION

We apply Theorem 2.1 to the study of invariants associated to sets of fat points in \mathbb{P}^n whose support is a complete intersection of points. For this section and the next, we shall assume that the field k is also algebraically closed.

Let P_1, \dots, P_s be s points in $\mathbb{P}^n := \mathbb{P}_k^n$. If m_1, \dots, m_s are s positive integers, then let $Z = \{(P_1, m_1), \dots, (P_s, m_s)\}$ denote the subscheme of \mathbb{P}^n defined by

$$I_Z = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_s^{m_s}$$

where \wp_i is the defining ideal in R of P_i . We call Z a *fat point scheme*, or a *set of fat points*. If $m_1 = \dots = m_s = m$, then we refer to Z as a *homogeneous scheme of fat points*, otherwise Z is *non-homogeneous*. The set $\text{Supp}(Z) = \{P_1, \dots, P_s\}$ is the support of Z .

If Z is a homogeneous fat point scheme of multiplicity m , then $I_Z = I_X^{(m)}$, the m th symbolic power of I_X where $X = \text{Supp}(Z)$. For an arbitrary support, $I_X^m \subset I_X^{(m)} = I_Z$ since I_X^m may not be saturated.

If $I = (F_1, \dots, F_n)$ is a complete intersection of type (d_1, \dots, d_n) , and if $I = \sqrt{I}$, then I is the defining ideal of $\prod_{i=1}^n d_i$ reduced points in \mathbb{P}^n . We denote this set by $X = CI(d_1, \dots, d_n)$ and call X a complete intersection. For each $m \in \mathbb{N}^+$ we write $Z = \{CI(d_1, \dots, d_n); m\}$ to denote the homogeneous fat point scheme of multiplicity m whose support is $CI(d_1, \dots, d_n)$. When $Z = \{CI(d_1, \dots, d_n); m\}$, then we have the equality $I_Z = I_X^{(m)} = I_X^m$ because of the following lemma:

Lemma 4.1 ([19, Lemma 5, Appendix 6]). *If I is a complete intersection, then $I^m = I^{(m)}$ for all positive integers m .*

Since the defining ideal of $\{CI(d_1, \dots, d_n); m\}$ is a power of a complete intersection, we therefore have

Proposition 4.2. *Suppose $Z = \{CI(d_1, \dots, d_n); m\} \subseteq \mathbb{P}^n$ with defining ideal I_Z . Then the graded minimal free resolution of I_Z is given by Theorem 2.1. The Hilbert function H_{R/I_Z} is given by Corollary 2.3.*

If Z is a fat point scheme with $\text{Supp}(Z) = CI(d_1, \dots, d_n)$, but Z is not homogeneous, then Proposition 4.2 can be used to obtain partial information about the invariants associated to Z . Recall that $\alpha(Z) := \min\{i \mid (I_Z)_i \neq 0\}$ and

$$\text{ri}(Z) := \min \left\{ i \mid H_{R/I_Z}(i) = \deg(Z) := \sum_{i=1}^s \binom{n + m_i - 1}{n} \right\}.$$

The invariant $\text{ri}(Z)$ is the regularity index of Z , while $\alpha(Z)$ is the smallest degree of a form contained in I_Z .

Proposition 4.3. *Let $Z = \{(P_1, m_1), \dots, (P_s, m_s)\} \subseteq \mathbb{P}^n$ be a non-homogeneous fat point scheme with $\text{Supp}(Z) = X = CI(d_1, \dots, d_n)$. Set $M = \max\{m_i\}_{i=1}^s$ and $m = \min\{m_i\}_{i=1}^s$. Then*

$$\sum_{\mathcal{L}_{n,m}} H_{R/I_X}(i - a_1 d_1 - \dots - a_n d_n) \leq H_{R/I_Z}(i) \leq \sum_{\mathcal{L}_{n,M}} H_{R/I_X}(i - a_1 d_1 - \dots - a_n d_n).$$

In particular

- (i) $md_1 \leq \alpha(Z) \leq Md_1$.
- (ii) $d_1 + \dots + d_{n-1} + md_n - n \leq \text{ri}(Z) \leq d_1 + \dots + d_{n-1} + Md_n - n$.

Proof. If I_X is the defining ideal of the support $CI(d_1, \dots, d_n)$, then we have

$$I_X^M \subseteq I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s} \subseteq I_X^m.$$

Since I_X^M and I_X^m define homogeneous fat point schemes on a complete intersection, the conclusions now follow from Proposition 4.2. \square

5. APPLICATION: FAT POINTS WITH SUPPORT ON A COMPLETE INTERSECTION MINUS A POINT

We discuss the invariants of fat point schemes whose support is a complete intersection minus a point. To provide some motivation, we recall that a set of points X has the Cayley-Bacharach property (CBP) if for every $P \in X$, $Y = X \setminus \{P\}$ always has the same Hilbert function. It is well known (see [10]) that a complete intersection of points satisfies the CBP because of the following result:

Theorem 5.1. *Let $X = CI(d_1, \dots, d_n) \subseteq \mathbb{P}^n$ be a complete intersection. Let $P \in X$ be any point, and set $Y = X \setminus \{P\}$. Then*

$$H_Y(i) = \min\{H_X(i), |X| - 1\} \text{ for all } i.$$

Moreover, because H_X depends only upon the type (d_1, \dots, d_n) , it follows that the Hilbert function of $Y = X \setminus \{P\}$ also depends upon the type.

Since the Hilbert function of $Z = \{CI(d_1, \dots, d_n); m\}$ depends upon the type and m , it is natural to wonder if a ‘‘Cayley-Bacharach like’’ result holds for Z , that is, if (P, m) is any fat point of Z , does the Hilbert function of $Y = Z \setminus \{(P, m)\}$ depend only upon the type and m ? As the next two examples

show, the answer is no since the construction of the underlying complete intersection must be taken into account. This suggests that it may be difficult to find general results to describe the invariants in this case.

Example 5.2. Let $X = CI(3, 4)$ be a complete intersection in \mathbb{P}^2 with defining ideal $I_X = (F, G)$. Set $Z = \{CI(3, 4); 3\}$. Take any point $P \in X = \text{Supp}(Z)$, and set $Y = Z \setminus \{(P, 3)\}$. If $F = L_1 L_2 L_3$ and $G = L'_1 L'_2 L'_3 L'_4$ are the product of linear forms, then the Hilbert function of Y is:

$$H_Y(i) : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 28 \ 36 \ 45 \ 54 \ 62 \ 65 \ 66 \rightarrow$$

On the other hand, if F and G are both irreducible, then H_Y is given by

$$H_Y(i) : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 28 \ 36 \ 45 \ 54 \ 62 \ 66 \rightarrow$$

The two Hilbert functions do not agree when $i = 11$.

Example 5.3. This example shows that, in some cases, the Hilbert function depends upon what point is removed from the underlying complete intersection. Let G_1, G_2 be two irreducible forms of $R = k[x, y, z]$ with $\deg G_1 = 2$ and $\deg G_2 = 3$. Let L_1 and L_2 be two linear forms that do not pass through the points in the complete intersection defined by (G_1, G_2) . Set $F_1 = L_1 G_1$ and $F_2 = L_2 G_2$, and let $X = CI(3, 4)$ denote the complete intersection of \mathbb{P}^2 defined by $I_X = (F_1, F_2)$. Let P_1 denote the point of X with defining ideal $I_{P_1} = (L_1, L_2)$, and let $P_2 \in X$ be a point that lies in the complete intersection defined by (G_1, G_2) . Let $Y_1 = \{X; 3\} \setminus \{(P_1, 3)\}$ and $Y_2 = \{X; 3\} \setminus \{(P_2, 3)\}$. We then have

$$\begin{aligned} H_{Y_1}(i) &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 28 \ 36 \ 45 \ 54 \ 62 \ 65 \ 66 \rightarrow \\ H_{Y_2}(i) &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 28 \ 36 \ 45 \ 54 \ 62 \ 66 \ 66 \rightarrow \end{aligned}$$

which fail to agree when $i = 11$.

Remark 5.4. In her Ph.D thesis [12] the first author introduced a generalization of the Cayley-Bacharach property for fat points. Specifically, a homogeneous set of fat points $Z \subseteq \mathbb{P}^n$ is said to satisfy the generalized Cayley-Bacharach property if for all fat points $(P, m) \in Z$, the set of fat points $Y = Z \setminus \{(P, m)\}$ has the same Hilbert function. As Example 5.3 shows, even if the support is a complete intersection, this generalized property need not hold.

As noted, these examples imply that extra hypotheses are needed on the support in order to study fat points whose support is a complete intersection minus a point. We therefore end this paper by considering $\alpha(Y)$ and $\text{ri}(Y)$ of $Y = \{X; m\} \setminus \{(P, m)\}$ when the support X “splits” into smaller complete intersections, say $X = X_1 \cup X_2$, with the property that the point P being removed from X belongs to X_2 and X_2 is contained in a hyperplane, i.e, $X_2 \subseteq H \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. This class of support allows us to make further use of Theorem 2.1. We introduce some relevant terminology.

Definition 5.5. The complete intersection $X = CI(d_1, \dots, d_n) \subseteq \mathbb{P}^n$ splits if X can be written as the disjoint union of two complete intersections:

$$X = CI(d_1, \dots, d_{i-1}, d_i - 1, d_{i+1}, \dots, d_n) \cup CI(d_1, \dots, d_{i-1}, 1, d_{i+1}, \dots, d_n)$$

for some $1 \leq i \leq n$ with $d_i \geq 2$.

Theorem 5.6. Let $X = CI(d_1, \dots, d_n) \subseteq \mathbb{P}^n$ with $d_1 \leq \dots \leq d_n$, and suppose that X splits as $X = CI(d_1, \dots, d_{n-1}, 1) \cup CI(d_1, \dots, d_{n-1}, d_n)$. Let $Z = \{X; m\}$ and $Y = \{X; m\} \setminus \{(P, m)\}$ for any $P \in CI(d_1, \dots, d_{n-1}, 1) \subseteq X$.

- (i) If $d_1 = d_n$, then $m(d_1 - 1) \leq \alpha(Y) \leq \alpha(Z) = md_1$.
- (ii) If $d_1 < d_n$, then $\alpha(Y) = \alpha(Z) = md_1$.

Proof. The complete intersection $CI(d_1, \dots, d_{n-1}, d_n - 1)$ is a subset of $\text{Supp}(Y) = X \setminus \{P\}$. Hence $Y' = \{CI(d_1, \dots, d_{n-1}, d_n - 1); m\}$ is a subscheme of Y . This implies $I_Y \subseteq I_{Y'}$, and thus $\alpha(Y') \leq \alpha(Y)$. But since the support of Y' is a complete intersection, by Theorem 4.2 we have $\alpha(Y') = m \cdot \min\{d_1, \dots, d_n - 1\}$. Because $d_1 \leq \dots \leq d_n$, if $d_1 = d_n$, then $\min\{d_1, \dots, d_n - 1\} = d_n - 1$, thus proving (i). In the case of (ii), $\min\{d_1, \dots, d_n - 1\} = d_1$. \square

If $Z \subseteq \mathbb{P}^n$ is any fat point scheme, and if $Y \subseteq Z$, then Lemma 1.1 of [18] implies $\text{ri}(Y) \leq \text{ri}(Z)$. Thus, if $Y \subseteq Z = \{CI(d_1, \dots, d_n); m\}$, we have by Theorem 4.2

$$\text{ri}(Y) \leq \text{ri}(Z) = d_1 + \dots + d_{n-1} + md_n - n.$$

Proposition 5.7. Let $X = CI(d_1, \dots, d_n) \subseteq \mathbb{P}^n$ with $2 \leq d_1 \leq \dots \leq d_n$ and suppose X splits as $X = X_1 \cup X_2 = CI(d_1 - 1, d_2, \dots, d_n) \cup CI(1, d_2, \dots, d_n)$. If $Z = \{X; m\}$, and $Y = Z \setminus \{(P, m)\}$ with $P \in X_2$, then

$$\text{ri}(Y) \geq d_1 + \dots + d_{n-1} + md_n - (n + 1).$$

Proof. Note that $Z = Z_1 \cup Z_2$ where $Z_1 = \{X_1; m\}$ and $Z_2 = \{X_2; m\}$. Hence $Y = Z_1 \cup Y_2$ where $Y_2 = \{X_2; m\} \setminus \{(P, m)\}$. From the short exact sequence

$$0 \rightarrow R/I_Y \rightarrow R/I_{Z_1} \oplus R/I_{Y_2} \rightarrow R/(I_{Z_1} + I_{Y_2}) \rightarrow 0$$

it follows that $H_Y(i) \leq H_{Z_1}(i) + H_{Y_2}(i)$ for all $i \in \mathbb{N}$. Set $r = d_1 + \dots + d_{n-1} + md_n - (n + 2)$. Then $H_{Z_1}(r) < \deg Z_1 = (d_1 - 1)d_2 \dots d_n \binom{m+n-1}{n}$ by Theorem 4.2 since Z_1 is a homogeneous fat point scheme on a complete intersection. Hence $H_Y(r) < \deg Z_1 + \deg Y_2 = \deg Y$, and thus $\text{ri}(Y) > r$. \square

Under the hypotheses of the previous proposition we have

$$d - (n + 1) \leq \text{ri}(Y) \leq d - n$$

where $d = d_1 + \dots + d_{n-1} + md_n$. If $X = CI(d_1, d_2) \subseteq \mathbb{P}^2$, we can give an exact formula when the support splits nicely. If $X \subseteq \mathbb{P}^2$ is any set of points, following [18], we define $b(X) := \min\{t \mid I_t \text{ contains a regular sequence of length two}\}$. A bound on the regularity index for any set of fat points in \mathbb{P}^2 is then given by

Theorem 5.8 ([18, Theorem 3.2]). *Let $X = \{P_1, \dots, P_s\}$ be a set of points in \mathbb{P}^2 with $m_1 \geq \dots \geq m_s$ a sequence of positive integers. For $i = 1, \dots, m_1$, let $Y_i = \{P_j \mid m_j \geq i\}$. If $Z = \{(P_1, m_1), \dots, (P_s, m_s)\}$, then*

$$\text{ri}(Z) \leq \text{ri}(X) + \sum_{i=2}^{m_1} b(Y_i).$$

If $X = CI(d_1, d_2)$, then $b(X) = d_2$. Furthermore, since $(I_X)_t = (I_Y)_t$ for $t \leq d_1 + d_2 - 3$ by Theorem 5.1, we also have $b(Y) = d_2$.

Corollary 5.9. *Let $X = CI(d_1, d_2) \subseteq \mathbb{P}^2$ with $2 \leq d_1 \leq d_2$ and suppose X splits as $X = X_1 \cup X_2 = CI(d_1 - 1, d_2) \cup CI(1, d_2)$. If $Z = \{X; m\}$, and $Y = Z \setminus \{(P, m)\}$ with $P \in X_2$, then*

$$\text{ri}(Y) = d_1 + md_2 - 3.$$

Proof. For $i = 1, \dots, m$, $Y_i = X \setminus \{P\}$ and therefore, $b(Y_i) = d_2$. The formula of Theorem 5.8 and the fact that $\text{ri}(X \setminus \{P\}) = d_1 + d_2 - 3$ implies $\text{ri}(Y) \leq d_1 + md_2 - 3$. The reverse inequality is Proposition 5.7. \square

Remark 5.10. If $I_X = (F_1, \dots, F_n)$ is the defining ideal of the complete intersection $X = CI(d_1, \dots, d_n) \subseteq \mathbb{P}^n$, and if each F_i is the product of d_i distinct linear forms, then the points of X form a n -dimensional “box”. For each $i = 1, \dots, n$, X splits as

$$X = CI(d_1, \dots, d_i - 1, \dots, d_n) \cup CI(d_1, \dots, 1, \dots, d_n).$$

Hence, the previous results about fat points on complete intersections which split apply to these configurations.

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